Winter School in Abstract Analysis 2024

Generalized Krom spaces and the Menger game

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1 Infinite games

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Definition

An *infinite game* is a pair $\mathrm{G} = (T, A)$ with $T \subset M^{<\omega}$ and $A \subset M^{\omega}$ for some set M such that

(I) If $t \in T$, then $t \restriction k \in T$ for all $k \leq |t|$;

(II) For all $t \in T$ there is an $x \in M$ such that $t^\frown x \in T$;

(III) $A \subset \mathrm{Runs}(G) = \{R \in M^{\omega} : R \restriction n \in T \text{ for all } n \in \omega\}$.

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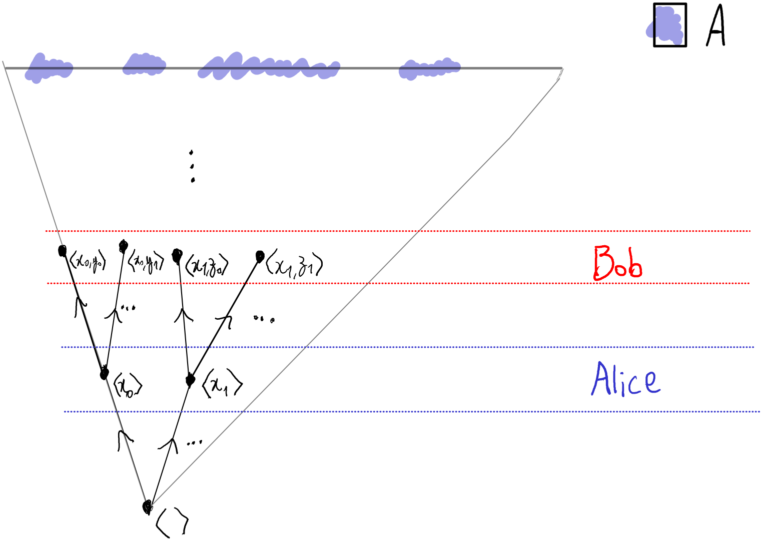
All of our games will be infinite in this talk, so we will omit the word "infinite" from now on.


A
Bob
Alice

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- If $t \in T$ and $|t|$ is even, then we say that it is $\textsc{Alice}$'s *turn* and $\left\{ x \in M : t^\frown x \in T \right\}$ is the set of all valid choices that $\textsc{Alice}$ can make at t.

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- If $t \in T$ and $|t|$ is odd, then we say that it is $\textsc{Bob}$'s *turn* and $\left\{ x \in M : t^\frown x \in T \right\}$ is the set of all valid choices that $\textsc{Bob}$ can make at t.

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- If $t \in T$ and $|t|$ is even, then we say that it is ALICE's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that ALICE can make at t.
- If $t \in T$ and $|t|$ is odd, then we say that it is BOB's *turn* and $\{ x \in M : t^\frown x \in T \}$ is the set of all valid choices that BOB can make at t.
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- If $t \in T$ and $|t| = 2n$ or $|t| = 2n + 1$, then we say that t is at the *nth inning*.
- We say that A is the *payoff set* of G: a run R is won by ALICE if $R \in A$ (and won by BOB otherwise).

Why *Alice* and *Bob*?

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Alice and Bob

24 languages

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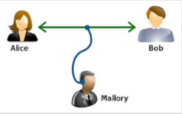
From Wikipedia, the free encyclopedia

Alice and Bob are fictional characters commonly used as placeholders in discussions about cryptographic systems and protocols,[1] and in other science and engineering literature where there are several participants in a thought experiment. The Alice and Bob characters were invented by Ron Rivest, Adi Shamir, and Leonard Adleman in their 1978 paper "A Method for Obtaining Digital Signatures and Public-key Cryptosystems".[2] Subsequently, they have become common archetypes in many scientific and engineering fields, such as quantum cryptography, game theory and physics.[3] As the use of Alice and Bob became more widespread, additional characters were added, sometimes each with a particular meaning. These characters do not have to refer to people; they refer to generic agents which might be different computers or even different programs running on a single computer.



Example scenario where communication between Alice and Bob is intercepted by Mallory

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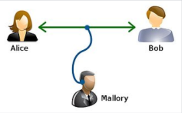

Example scenario where communication between Alice and Bob is intercepted by Mallory

Although Alice and Bob were invented with no reference to their personality, authors soon began adding colorful descriptions. In 1983, Blum invented a backstory about a troubled relationship between Alice and Bob, writing, "Alice and Bob, recently divorced, mutually distrustful, still do business together. They live on opposite coasts, communicate mainly by telephone, and use their computers to transact business over the telephone."[9] In 1984, John Gordon delivered his famous[10] "After Dinner Speech" about Alice and Bob, which he imagines to be the first "definitive biography of Alice and Bob."[11]

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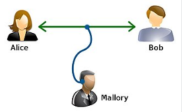

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Cast of characters [edit]

The most common characters are Alice and Bob. Eve, Mallory, and Trent are also common names, and have fairly well-established "personalities" (or functions). The names often use alliterative mnemonics (for example, Eve, "eavesdropper"; Mallory, "malicious") where different players have different motives. Other names are much less common and more flexible in use. Sometimes the genders are alternated: Alice, Bob, Carol, Dave, Eve, etc.[14]

Example (Banach-Mazur game)

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  - ALICE chooses a non-empty open set U_n contained in the open set V_{n-1} chosen by BOB in the previous inning;
  - BOB responds with a non-empty open set $V_n \subset U_n$.

Then BOB wins the run $\langle U_0, V_0, \ldots, U_n, V_n, \ldots \rangle$ if $\bigcap_{n \in \omega} V_n \neq \emptyset$ (and ALICE wins otherwise).

Definition

A space X is Baire if for every sequence $\langle A_n : n \in \omega \rangle$ of dense open sets of X, $\bigcap_{n\in\omega} A_n$ is dense in X.

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Theorem (Oxtoby – 1957)

A nonempty space X is Baire if, and only if, $\mathrm{A} \not\uparrow \mathsf{BM}(X)$.

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Then Bob wins the run $\langle \mathcal{U}_0, \mathcal{F}_0, \ldots, \mathcal{U}_n, \mathcal{F}_n, \ldots \rangle$ if $\bigcup_{n\in\omega} \mathcal{F}_n$ is an open cover for X (and Alice wins otherwise).

Definition

A topological space X is *Menger* if for every sequence of open covers $\langle \mathcal{U}_n : n \in \omega \rangle$ there is a sequence $\langle \mathcal{F}_n : n \in \omega \rangle$ such that $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ for every $n \in \omega$ and $\bigcup_{n\in\omega} \mathcal{F}_n$ is an open cover for X.

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Theorem (Hurewicz – 1926)

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In this case, we consider the ultrametric over $\mathrm{K}(X)$ defined by

$$d(R,S) = \begin{cases} \frac{1}{\Delta(R,S)+1}, & \text{if } R \neq S \\ 0, & \text{otherwise,} \end{cases}$$

where $\Delta(R,S) = \min\{\, n \in \omega : R(n) \neq S(n) \,\}$.

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Remark

Note that $R \in \mathrm{K}(X)$ if, and only if, R is a run of $\mathrm{BM}(X)$ in which Bob wins!

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Obviously, $\mathsf{K}(X) = \mathsf{K}(\mathsf{BM}(X))$ for every nonempty space X.

Definition

Let X be a nonempty space and suppose $\mathcal{B}$ is a basis for X. We denote by $\mathrm{BM}(X, \mathcal{B})$ the game played as in the Banach-Mazur game with the added restriction that both players must choose open sets exclusively from $\mathcal{B}$.

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Fact

The games $\mathsf{BM}(X,\mathcal{B})$ *and* $\mathsf{BM}(X)$ *are equivalent, that is,*

$$\mathrm{A}\uparrow\mathsf{BM}(X,\mathcal{B}) \iff \mathrm{A}\uparrow\mathsf{BM}(X),$$
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$$\mathrm{B} \uparrow \mathsf{BM}(X, \mathcal{B}) \iff \mathrm{B} \uparrow \mathsf{BM}(X).$$

So, given $G = (T, A)$, we will only consider the moves made in $\mathsf{BM}(\mathsf{K}(G))$ of the form

$$[t] = \{\, R \in \mathsf{K}(G) : R \text{ extends } t \,\},$$

with $t \in T$.

Let us recall the following theorem from Group Theory, which states that symmetric groups are, in some sense, "universal":

Theorem (Cayley – 1854)

For every group G there is a set $X(G)$ such that G is isomorphic to a subgroup of the symmetric group of $X(G)$.

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We also have the "universality" of the Banach-Mazur game:

Theorem (D., Szeptycki, Tholen – 2024)

For every game G there is a metrizable space $\mathrm{K}^*(G)$ such that G is isomorphic to a subgame of the Banach-Mazur game over $\mathrm{K}^*(G)$.

But how different can G be from BM(K(G))?

Example

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Example

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Example

Let X be a space such that $\mathrm{B} \not\uparrow \mathsf{Menger}(X)$. Note that $\mathrm{B} \uparrow \mathsf{BM}(\mathsf{K}(\mathsf{Menger}(X)))$ (trivially!):

- Suppose ALICE begins by choosing a basic open set identified by $\langle \mathcal{U}_0, \mathcal{F}_0, \ldots, \mathcal{U}_n \rangle$.
- Then BOB can respond with $\langle \mathcal{U}_0, \mathcal{F}_0, \ldots, \mathcal{U}_n \rangle ^\frown \langle \mathcal{F}_n, \{X\} \rangle$ and game over.

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Theorem

For every nonempty space X, $\mathrm{BM}(X)$ *is equivalent to* $\mathrm{BM}(\mathrm{K}(\mathrm{BM}(X)))$.

Definition

For a space X, let Menger$^*(X)$ denote the game played exactly as the Menger game over X, with the new restriction stating that ALICE must choose in the inning $n+1$ an open cover which refines the open cover that she chose in the nth inning.

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Fact

For every space X*, the game* $\mathrm{Menger}^*(X)$ *is equivalent to* $\mathrm{Menger}(X)$*.*

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Corollary

A space X *is Menger if, and only if,* $\mathrm{K}(\mathrm{Menger}^*(X))$ *is Baire.*

Idea of the theorem's proof:

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blackboard!

Referências

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Děkuji!

Thank you!